

Eigenvalues of Fokker-Planck operators

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1980 J. Phys. A: Math. Gen. 13 2599

(<http://iopscience.iop.org/0305-4470/13/8/008>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 129.252.86.83

The article was downloaded on 30/05/2010 at 17:42

Please note that [terms and conditions apply](#).

Eigenvalues of Fokker–Planck operators

Ivan Kuščer† and Reinhard Illner

Fachbereich Mathematik, Universität Kaiserslautern D-6750 Kaiserslautern, Federal Republic of Germany

Received 5 June 1979

Abstract. Fourier-transformed Fokker–Planck transport operators, $T(\mathbf{k}) = -(\nabla - \mathbf{v}) \cdot \zeta(\mathbf{v}) \cdot \nabla + i\mathbf{k} \cdot \mathbf{v}$, have discrete spectra for most physically reasonable friction tensors $\zeta(\mathbf{v})$. The set of eigenvalues $\{\lambda_\alpha(k)\}$ represents a collection of single and/or multiple-valued functions analytic in k , with no other singularities than branch points, which are also branch points for the eigenprojections $P_\alpha(k)$. Corresponding analytic behaviour is found with the eigenfunctions $\phi_\alpha(\mathbf{k}, \mathbf{v})$, chosen in such a way that $(\phi_\alpha, \phi_\beta^*) = \delta_{\alpha\beta}$ and $P_\alpha = (\cdot, \phi_\alpha^*)\phi_\alpha$. Eigennilpotents and generalised eigenfunctions (which are not proper eigenfunctions) only appear at branch points and possibly at some other meeting points of eigenvalues. Each eigenvalue remains real for sufficiently small real k , until it meets its first branch point. Within this region, the eigenfunctions are conveniently classified by three labels analogous to quantum numbers, $\alpha = (n, l, m)$. A necessary condition for the meeting point of two eigenvalues λ_{nlm} and $\lambda_{n'l'm'}$ to be a branch point is $m = m'$.

1. Introduction

In Markovian approximations to linearised kinetic theory, one-particle distributions are sought as solutions of various kinetic equations. If no internal degrees of freedom are involved, we write

$$f(\mathbf{r}, \mathbf{v}, t) = (2\pi)^{-3/2} \exp(-v^2/2) [\text{const} + h(\mathbf{r}, \mathbf{v}, t)],$$

where v^2 stands for mv^2/kT . In absence of sources and external forces, the kinetic equation for h has the form

$$(\partial/\partial t + \mathbf{v} \cdot \nabla + C)h(\mathbf{r}, \mathbf{v}, t) = 0, \quad (1)$$

where C is the collision operator. We recall (Illner and Kuščer 1979, henceforth referred to as I) that C is the generator of a semigroup describing the Markov process under consideration. The ansatz

$$h(\mathbf{r}, \mathbf{v}, t) = \phi_\alpha(\mathbf{k}, \mathbf{v}) \exp(i\mathbf{k} \cdot \mathbf{r}) \exp(-\lambda_\alpha t)$$

describes distributions periodic in space and approaching equilibrium exponentially. This leads to the eigenvalue problem

$$(T(\mathbf{k}) - \lambda_\alpha)\phi_\alpha = 0, \quad (2)$$

with the operator $T(\mathbf{k}) = C + i\mathbf{k} \cdot \mathbf{v}$.

† Now at the Department of Physics, University of Ljubljana, PO Box 543, Ljubljana, Yugoslavia.

Diffusion (Brownian motion) of a particle weakly coupled to the surrounding medium or of isolated heavy particles in a gas of light molecules (Rayleigh gas mixture) can be approximated by a continuous Markov process. The same holds for any combination of both cases, eg for the diffusion of isolated heavy ions in a light plasma. The kinetic equation for such a process is of the Fokker–Planck type, with the collision operator having the form

$$C = -(\nabla - \mathbf{v}) \cdot \boldsymbol{\zeta}(\mathbf{v}) \cdot \nabla,$$

or in spherical coordinates (Corngold 1977)

$$C = -\frac{1}{v^2} \left[\left(\frac{\partial}{\partial v} - v \right) v^2 \zeta_{\parallel}(v) \frac{\partial}{\partial v} + \zeta_{\perp}(v) \left(\frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \sin \vartheta \frac{\partial}{\partial \vartheta} + \frac{1}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2} \right) \right], \quad (3)$$

where $\boldsymbol{\zeta}(\mathbf{v})$ is the friction tensor and $\zeta_{\parallel}(v)$ and $\zeta_{\perp}(v)$ its components parallel and perpendicular to \mathbf{v} .

In an earlier paper (I) the spectra of C and $T(\mathbf{k})$ were investigated in the space $L^2(\mathbb{R}^3)$ with Maxwellian weight. $T(0) = C$ is selfadjoint in this space. It was found that for a large class of friction tensors both spectra are discrete, which is to say that the resolvent $(T(\mathbf{k}) - \lambda)^{-1}$ is compact for λ not in the spectrum of $T(\mathbf{k})$. Sufficient conditions for this to be true are that the tensor $\boldsymbol{\zeta}(\mathbf{v})$ is positive-definite, that its components are locally bounded and once continuously differentiable, and that for $v \rightarrow \infty$

$$\lim \zeta_{\parallel}(v) > 0, \quad \lim(1/v\zeta_{\parallel}(v))(d\zeta_{\parallel}/dv) = 0.$$

This paper is intended to analyse the general behaviour of eigenvalues and eigenfunctions when \mathbf{k} is varied. The properly extended $T(\mathbf{k})$ will be regarded as a member of a holomorphic family of closed form-sectorial operators with common domain $D(C)$ (I), ie, a family of type B according to Kato (1966, henceforth referred to as K). Rotational invariance of C makes sure that the eigenvalues only depend upon $k^2 = k_x^2 + k_y^2 + k_z^2$. For real k they are real or they appear in complex-conjugate pairs (I). The operator C is real in the sense that it makes real functions out of real ones. Therefore, if $\phi_{\alpha}(\mathbf{k}, v)$ is an eigenfunction belonging to the eigenvalue $\lambda_{\alpha}(k)$ of $T(\mathbf{k})$, then $\phi_{\alpha}^*(\mathbf{k}, v)$ belongs to the eigenvalue $\lambda_{\alpha}^*(k)$ of $T^{\dagger}(\mathbf{k}) = T(-\mathbf{k}^*)$.

The uncomplicated picture of the eigenvalues, eigenprojections and eigenfunctions which will emerge from the subsequent analysis is characteristic of operators of the form $C + ikL$, where C is real and positive-semidefinite and has a discrete spectrum, while L is real and selfadjoint, with the form $(L\chi, \chi)$ having a vanishing relative bound with respect to $(C\chi, \chi)$.

2. Branching of eigenvalues

We are going to investigate the $T(\mathbf{k})$ for $\mathbf{k} = k\mathbf{e}$, where \mathbf{e} shall be a fixed real unit vector, while k varies and may be complex. The restriction yields a one-parametric family of operators, which is holomorphic in the whole complex plane of k . Such a family is known to have the following properties (see K, § II.1, § VII.1.3, Theorem 1.8 and subsequent remark):

For any point k_0 and for any finite collection of eigenvalues $\{\lambda_{\alpha}(k)\}$ we can find an open neighbourhood of k_0 where $\{\lambda_{\alpha}(k)\}$ constitutes a set of single and/or multiple-valued analytic functions, with no other singularities except possibly a branch point at

k_0 . Even then, they remain continuous at k_0 . Branch points can only, but do not necessarily, occur at those values of k where two or more of the eigenvalues meet.

If two eigenvalues $\lambda_\alpha(k)$ and $\lambda_\beta(k)$ meet at a ‘meeting point’ k_0 , both may be analytically continued around this point. Upon completion of the cycle, the original eigenvalues are obtained either in the original or transposed order. In the first case, $\lambda_\alpha(k)$ and $\lambda_\beta(k)$ represent analytic functions which are distinct in the neighbourhood of k_0 ; the curves $\lambda_\alpha(k)$ and $\lambda_\beta(k)$ just cross, without affecting each other. In the second case k_0 is a branch point of first order, and the Puiseux series begins as follows (K, § II.1.2),

$$\lambda_{\alpha,\beta}(k) = \lambda(k_0) \pm c_1(k_0 - k)^{1/2} + c_2(k_0 - k) \pm c_3(k_0 - k)^{3/2} + \dots \quad (4)$$

According to the theory of selfadjoint holomorphic families of operators (K, § VII.3.1), each eigenvalue $\lambda_\alpha(k)$ is a holomorphic function in some neighbourhood of $k = 0$, where $T(k)$ is selfadjoint. None of the eigenvalues can have a branch point there. Next we consider a varying real k , starting with $k = 0$. If some $\lambda(0)$ is degenerate, it may split into several eigenvalues $\lambda_\alpha(k), \lambda_\beta(k), \dots$, each behaving as a distinct analytic function near $k = 0$. Each is even in k and real for purely imaginary k since in this case $T(k)$ is selfadjoint. That is, each admits a series in powers of k^2 with real coefficients. Consequently, also for real k each $\lambda_\alpha(k)$ remains real for a while, namely until it meets another eigenvalue at some k_0 such that this is the first branch point for $\lambda_\alpha(k)$. If only two eigenvalues meet there, they are real below k_0 and complex-conjugate beyond, according to equation (4).

3. Eigenprojections

Along with the eigenvalues $\lambda_\alpha(k)$ also the corresponding (non-selfadjoint) eigenprojections $P_\alpha(k)$ and eigennilpotents $D_\alpha(k)$ are holomorphic functions in some neighbourhood of the chosen k , except possibly at meeting points (K, § II.1.5). Through analytic continuation around such points we recognise the projections as single- or multiple-valued analytic functions. On the other hand, all the nilpotents vanish everywhere except possibly at meeting points. This is seen by first considering a purely imaginary k , when $T(k)$ is selfadjoint so that it does not contain any nilpotents. The statement then follows by analytic continuation.

Let us consider an eigenvalue $\lambda(0)$ which for $k \neq 0$ splits into several eigenvalues $\lambda_\alpha(k)$, $\alpha \in \{\alpha\}$, some of which may be permanently degenerate. The chosen k shall not be a meeting point for any member of the set $\{\lambda_\alpha(k)\}$. Let $P_\alpha(k)$ be the total eigenprojection associated with a particular $\lambda_\alpha(k)$. Suppressing the variable v we denote by $\phi_{\alpha j}(k)$ the corresponding linearly independent eigenfunctions ($j = 1, 2, \dots, r$ for an r -fold λ_α). Only proper eigenfunctions appear because eigennilpotents are absent.

In going from 0 to k we take a path in the complex plane that avoids all meeting points of the $\lambda_\alpha(k)$, except for the splitting at $k = 0$. Thereby a bounded invertible transformation $U(k)$ is defined, which depends analytically on k along the chosen path and obeys the set of equations (K, § II.4.5, see also §§ II.4.2 and VII.1.3)

$$U^{-1}(k)P_\alpha(k) = P_\alpha(0)U^{-1}(k), \quad \alpha \in \{\alpha\}.$$

Applying both sides to $\phi_{\alpha j}(k)$ we notice that the function $\phi_{\alpha j}(0) := U^{-1}(k)\phi_{\alpha j}(k)$ obeys

the relation

$$\phi_{\alpha j}(0) = P_{\alpha}(0)\phi_{\alpha j}(0).$$

Hence this is an eigenfunction corresponding to $\lambda(0)$. Linear independence is preserved by the transformation. The transformation operator $U(\mathbf{k})$ is the solution of the differential equation

$$U'(\mathbf{k}) = Q(\mathbf{k})U(\mathbf{k}), \quad \text{with} \quad U(0) = 1,$$

$$Q(\mathbf{k}) = \sum_{\alpha} P'_{\alpha}(\mathbf{k})P_{\alpha}(\mathbf{k}) - \sum_{\alpha} P'_{\alpha}(\mathbf{k}) \left[1 - \sum_{\alpha} P_{\alpha}(\mathbf{k}) \right].$$

Next we join the points 0 and k^* by a path symmetric to the one we took from 0 to k . The transformation $U(-\mathbf{k}^*)$ yields eigenfunctions corresponding to the eigenvalue $\lambda_{\alpha}(k^*) = \lambda_{\alpha}^*(k)$ of $T(-\mathbf{k}^*)$,

$$\phi_{\alpha j}(-\mathbf{k}^*) := U(-\mathbf{k}^*)\phi_{\alpha j}(0) = U(-\mathbf{k}^*)U^{-1}(\mathbf{k})\phi_{\alpha j}(\mathbf{k}). \quad (5a)$$

However, for the same eigenvalue we already have the eigenfunctions $\phi_{\alpha j}^*(\mathbf{k})$, which must be linear combinations of the $\phi_{\alpha j}(-\mathbf{k}^*)$.

Further conclusions can be drawn from the reality of the operator C . A consequence is that $T(\mathbf{k})$ and $T(-\mathbf{k}^*) = T^{\dagger}(\mathbf{k})$ are complex conjugates of each other, in the sense that $(T(\mathbf{k})\chi)^* = T(-\mathbf{k}^*)\chi^*$. We are going to write $T^*(\mathbf{k}) = T(-\mathbf{k}^*)$. In the same sense we may state that $P_{\alpha}^*(\mathbf{k}) = P_{\alpha}(-\mathbf{k}^*)$ and $U^*(\mathbf{k}) = U(-\mathbf{k}^*)$. The first assertion follows from the definition of P_{α} in terms of a Cauchy integral (K, § I.5.3), while the second is obtained by substituting $-\mathbf{k}^*$ for \mathbf{k} into the above differential equation for $U(\mathbf{k})$. Thus the complex conjugate to relation (5a) is

$$\phi_{\alpha j}^*(-\mathbf{k}^*) = U(\mathbf{k})\phi_{\alpha j}^*(0) = U(\mathbf{k})U^{-1}(-\mathbf{k}^*)\phi_{\alpha j}^*(\mathbf{k}),$$

or

$$\phi_{\alpha j}^*(\mathbf{k}) = U(-\mathbf{k}^*)\phi_{\alpha j}^*(0) = U(-\mathbf{k}^*)U^{-1}(\mathbf{k})\phi_{\alpha j}^*(-\mathbf{k}^*). \quad (5b)$$

For each j we now choose one of the combinations $\phi_{\alpha j}(\mathbf{k}) + \phi_{\alpha j}^*(-\mathbf{k}^*)$ or $i[\phi_{\alpha j}(\mathbf{k}) - \phi_{\alpha j}^*(-\mathbf{k}^*)]$ in such a way that again a linearly independent set of eigenfunctions corresponding to $\lambda_{\alpha}(k)$ is obtained. (One can show that this is always possible.) These new eigenfunctions will henceforth be denoted as $\phi_{\alpha j}(\mathbf{k})$. A combination of equations (5a) and (5b) yields $\phi_{\alpha j}^*(\mathbf{k})$ as the adjoint solutions. Moreover, a unique set $\phi_{\alpha j}(0)$ of real eigenfunctions corresponding to $\lambda(0)$ is obtained from both sides,

$$\phi_{\alpha j}(0) = U^{-1}(\mathbf{k})\phi_{\alpha j}(\mathbf{k}) = U^{-1}(-\mathbf{k}^*)\phi_{\alpha j}^*(\mathbf{k}). \quad (5c)$$

We orthogonalise and normalise the set thus obtained, so that henceforth $(\phi_{\alpha i}(0), \phi_{\alpha j}(0)) = \delta_{ij}$. Substituting expressions (5c) and taking account of the identity $U^{-1}(-\mathbf{k}^*) = U^{\dagger}(\mathbf{k})$ (K, § II.6.2), we find that for $k \neq 0$ the eigenfunctions and their adjoints, redefined according to equation (5c), constitute a biorthonormal set,

$$(\phi_{\alpha i}(\mathbf{k}), \phi_{\beta j}^*(\mathbf{k})) = \delta_{\alpha\beta}\delta_{ij}. \quad (6)$$

For $\lambda_{\alpha} \neq \lambda_{\beta}$ this relation follows directly from equation (2).

Since $U(\mathbf{k})$ depends upon the chosen path, so do the $\phi_{\alpha j}(\mathbf{k})$ for given $\phi_{\alpha j}(0)$. Whenever the path is moved to the other side of a branch point of $\lambda_{\alpha}(k)$, the $\phi_{\alpha j}(\mathbf{k})$ change.

To simplify the notation we shall henceforth drop the label j , so that α will be used to specify the individual biorthonormalised eigenfunctions. The one-dimensional projection associated with $\phi_\alpha(\mathbf{k})$ is described by

$$P_\alpha \chi = (\chi, \phi_\alpha^*) \phi_\alpha, \tag{7}$$

as can be inferred from the commutation property (K, § III.5.6), $P_\alpha(\mathbf{k})T(\mathbf{k})\chi = T(\mathbf{k})P_\alpha(\mathbf{k})\chi$ for all $\chi \in D(C)$, and with the aid of equation (6) in connection with the requirement that $P_\alpha^2 = P_\alpha$ and $P_\alpha P_\beta = 0$. We conclude that

$$\|P_\alpha\| = \|\phi_\alpha\|^2 \geq 1. \tag{8}$$

If at some meeting point this norm diverges, an eigenfunction can still be constructed by renormalising ϕ_α , as we shall see later.

4. Branch points

To understand the situation at branch points, it will suffice to examine the case of two simple eigenvalues $\lambda_\alpha(k)$ and $\lambda_\beta(k)$ meeting at their branch point k_0 . We already have expansion (4) for the eigenvalues, where we assume that $c_1 \neq 0$. Similarly, the projections should expand as (K, § II.1, Theorem 1.9)

$$P_{\alpha,\beta}(\mathbf{k}) = \pm D(k_0 - k)^{-1/2} + \frac{1}{2}P + O[(k_0 - k)^{1/2}], \tag{9}$$

where P is the total projection corresponding to $\lambda(k_0)$. Since $P_{\alpha,\beta}^2 = P_{\alpha,\beta}$ and $P_\alpha P_\beta = 0$, D must be a nilpotent, such that $D^2 = 0$ and $PD = DP = D$. Next we observe that

$$\lambda_\alpha P_\alpha + \lambda_\beta P_\beta = \lambda(k_0)P + 2c_1 D + O[(k_0 - k)^{1/2}], \tag{10}$$

which in the limit gives the spectral resolution of $T(k_0)P$ (K, § III.6.5). Terms diverging at $k \rightarrow k_0$ would enter this result if expansion (9) were to include any higher negative powers of $(k_0 - k)$. Such powers are ruled out thereby, so that the expansion is confirmed.

The square roots of the form $(P_\alpha \chi^*, \chi) = (\phi_\alpha, \chi)^2$ and of a similar one for P_β , with equation (9) substituted, show that the Puiseux expansions for the eigenfunctions must have the form

$$\phi_\alpha(\mathbf{k}, \mathbf{v}) = \phi(\mathbf{v})(k_0 - k)^{-1/4} + \frac{1}{2}\psi(\mathbf{v})(k_0 - k)^{1/4} + \dots \tag{11a}$$

$$\phi_\beta(\mathbf{k}, \mathbf{v}) = -i\phi(\mathbf{v})(k_0 - k)^{-1/4} + \frac{1}{2}i\psi(\mathbf{v})(k_0 - k)^{1/4} + \dots \tag{11b}$$

According to equations (7) and (9) the coefficients ϕ and ψ are such that

$$D\chi = (\chi, \phi^*)\phi, \quad P\chi = (\chi, \psi^*)\phi + (\chi, \phi^*)\psi. \tag{12}$$

When k cycles around the branch point k_0 , the eigenfunction ϕ_α consecutively turns into $\phi_\beta, -\phi_\alpha, -\phi_\beta$, as seen from equation (11a, b). Since the sign is irrelevant, only two of the four branches are meaningful.

For P and D to have the required properties, the function ϕ and ψ must obey the relations

$$(\phi, \phi^*) = 0, \quad (\phi, \psi^*) = 1, \quad (\psi, \psi^*) = 0. \tag{13}$$

Therefore

$$P\phi = \phi, \quad P\psi = \psi, \quad D\phi = 0, \quad D\psi = \phi, \tag{14}$$

as was to be expected. Letting $T(\mathbf{k}_0)P = \lambda(k_0)P + 2c_1D$ act upon ϕ , we see that this is the renormalised eigenfunction corresponding to $\lambda(k_0)$. The same procedure reveals that ψ is a generalised eigenfunction such that

$$(T(\mathbf{k}_0) - \lambda(k_0))\psi = 2c_1\phi, (T(\mathbf{k}_0) - \lambda(k_0))^2\psi = 0. \tag{15}$$

Thus the algebraic multiplicity of $\lambda(k_0)$ ($= 2$) exceeds the geometric multiplicity ($= 1$). The situation is in complete analogy with Kato's example involving a symmetric two by two matrix (\mathbf{K} , examples II.1.1a and 1.12a).

In passing we can take a look at the exceptional possibility that $c_1 = 0$ but, say, $c_3 \neq 0$ in equation (4). The expansions for $P_{\alpha,\beta}$ then begin with $\pm D(k_0 - k)^{-3/2}$ and that for ϕ_α with $\phi(k_0 - k)^{-3/4}$. Again $D = (\cdot, \phi^*)\phi$ and $D^2 = 0$.

If two simple eigenvalues remain holomorphic at a meeting point k_0 , so that this is not a branch point, it may still happen that the corresponding (renormalised) eigenfunctions merge. In such exceptional cases the projections $P_{\alpha,\beta}$ have poles at k_0 (\mathbf{K} , examples II.1.1f and 1.12f), and the expansions are

$$\lambda_{\alpha,\beta} = \lambda(k_0) + \lambda'_{\alpha,\beta}(k - k_0) + \dots \tag{16}$$

$$P_{\alpha,\beta} = \pm D(k - k_0)^{-1} + A_{\alpha,\beta} + \dots \tag{17}$$

$$\phi_\alpha = \phi(k - k_0)^{-1/2} + \psi_\alpha(k - k_0)^{1/2} + \dots \tag{18}$$

and similarly for ϕ_β . We have assumed that $\lambda'_\alpha \neq \lambda'_\beta$, so that higher negative powers in (17) and (18) are ruled out by a similar argument as before. The total projection corresponding to $\lambda(k_0)$ is $P = A_\alpha + A_\beta$ and the generalised eigenfunction $\psi = \psi_\alpha + \psi_\beta$. Relations of the form (12)–(15) again hold.

The results are similar for meeting points of permanently degenerate eigenvalues, except that the whole basis of eigenfunctions belonging to each eigenvalue must be taken into account. If more than two eigenvalues form a cycle at a common branch point, this affects the exponents in equations (4), (9) etc. in an obvious way.

Let us summarise: Eigennilpotents appear at branch points and possibly at some other meeting points of the eigenvalues. The eigenprojections are singular there. There and only there have eigenvalues greater algebraic than geometric multiplicity. In general, such points only occur at special values of k .

Any linear combination of eigenmodes $\phi_\alpha \exp(i\mathbf{k} \cdot \mathbf{r}) \exp(-\lambda_\alpha t)$ belonging to a multiple eigenvalue $\lambda_\alpha(k)$ represents, of course, an eigenmode decaying at the same rate. A natural source of such degeneracy is the rotational invariance of the operator involved (see § 6).

The merging of different modes at branch points gives rise to non-exponentially decaying modes involving generalised eigenfunctions. To show this, we consider the following limit,

$$\begin{aligned} h(\mathbf{r}, \mathbf{v}, t) &= \lim_{k \rightarrow k_0} (k_0 - k)^{-1/4} [\phi_\alpha \exp(-\lambda_\alpha t) - i\phi_\beta \exp(-\lambda_\beta t)] \exp(i\mathbf{k} \cdot \mathbf{r}) \\ &= [\psi(\mathbf{v}) - 2c_1\phi(\mathbf{v})t] \exp(i\mathbf{k}_0 \cdot \mathbf{r}) \exp(-\lambda t), \end{aligned} \tag{19}$$

where equations (4) and (11) have been taken into account. With the help of equation (15) we can verify directly that the resulting expression (19) is indeed a solution of the kinetic equation (1).

5. Boundedness of eigenvalues

In order to obtain a bound for the eigenvalues, we follow the reasoning applied by Kato (K, § VII.4.7) to self-adjoint families of operators of type B. In the standard fashion of perturbation theory we take the derivative of equation (2) with respect to k and then the scalar product with ϕ_α^* , to obtain

$$\lambda'_\alpha = (iv \cos \vartheta \phi_\alpha, \phi_\alpha^*). \quad (20)$$

where ϑ is the angle between the vectors k and v .

Next we vary k from 0 to some value k_1 along a path which avoids all singularities of P_α , in particular the branch points of λ_α , so that this eigenvalue is uniquely determined along the path. In any neighbourhood of this path, there is only a finite number of singularities of P_α . Indeed, an accumulation of such points would represent a stronger singularity and is ruled out by what has been covered in § 2. Thus the norm of the projection P_α corresponding to ϕ_α remains bounded along the path: $\|P_\alpha(k)\| = \|\phi_\alpha(k)\|^2 \leq M$.

It has been shown in I that the quadratic form $(v \cos \vartheta \chi, \chi)$ is relatively bounded with respect to $(C\chi, \chi)$, with a vanishing relative bound. The proof implies that the same is true for the form $(v |\cos \vartheta| \chi, \chi)$, ie: for any $b > 0$ an $a > 0$ can be found such that

$$(v |\cos \vartheta| \chi, \chi) \leq a \|\chi\|^2 + b(C\chi, \chi)$$

for all $\chi \in D(C)$. Substituting $C = T(k) - ikv \cos \vartheta$, we obtain a similar statement for $T(k)$ with bounded k , say $|k| \leq K$,

$$(v |\cos \vartheta| \chi, \chi) \leq a' \|\chi\|^2 + b' |(T(k)\chi, \chi)|,$$

where $a' = a/(1 - bK)$ and $b' = b/(1 - bK)$. We choose $b < K^{-1}$ to make a' and b' positive.

A sequence of inequalities now follows from equation (20),

$$\begin{aligned} |d\lambda_\alpha/dk| &= |(iv \cos \vartheta \phi_\alpha, \phi_\alpha^*)| \leq (v |\cos \vartheta| \phi_\alpha, \phi_\alpha) \\ &\leq a' \|\phi_\alpha\|^2 + b' |(T(k)\phi_\alpha, \phi_\alpha)| \leq (a' + b' |\lambda_\alpha|) M. \end{aligned}$$

Integration over k , with $|d\lambda_\alpha| \geq d|\lambda_\alpha|$ taken into account, yields the bound

$$a' + b' |\lambda_\alpha(k_1)| \leq (a' + b' |\lambda_\alpha(0)|) \exp(b' Ms), \quad (21)$$

where s is the length of the path from $k = 0$ to k_1 ,

$$s = \int_0^{k_1} |dk|.$$

The growth rate is not very rapid since b' can be made arbitrarily small, though at the expense of increasing a' .

The conclusion is that the eigenvalues remain finite at finite k . A running away to infinity, as is possible for eigenvalues of operator families having a parameter-dependent domain (K, examples V.4.14 and VII.1.11), cannot occur. Hence the complete set of eigenvalues of $T(k)$ represents a collection of analytic functions of k with no other singularities than branch points in the finite plane.

6. Classification of eigenfunctions

For the selfadjoint $T(0) = C$ one can derive, using expression (3), a complete set of eigenfunctions factorised in spherical coordinates,

$$\phi_{nlm}(0, \mathbf{v}) = u_{nl}(v) Y_{lm}(\vartheta, \varphi), \quad (22)$$

$$Y_{lm}(\vartheta, \varphi) = P_{lm}(\cos \vartheta) \begin{cases} \cos m\varphi \\ \sin |m|\varphi \end{cases},$$

$$n = 0, 1, 2, \dots; \quad l = n, n-2, n-4, \dots, 1 \text{ or } 0,$$

$$m = 0, \pm 1, \pm 2, \dots, \pm l.$$

Three labels n, l, m , analogous to quantum numbers for the isotropic three-dimensional oscillator (Morse and Feshbach 1953), are conveniently introduced for classification. The α or α_j etc. used before stand for triplets n, l, m . We have chosen to associate the factor $\cos m\varphi$ with $m \geq 0$, and $\sin |m|\varphi$ with $m < 0$.

Rotational invariance of C also implies that the eigenvalues only depend upon n and l ; hence $\lambda_{nl}(0)$ is at least $(2l+1)$ -fold. In the special case with constant isotropic ζ there is, in addition, an accidental degeneracy in l , so that the eigenvalues only depend upon n , and λ_n is $\frac{1}{2}(n+1)(n+2)$ -fold. This can be concluded from mathematical equivalence with the Schrödinger equation for the 3D harmonic oscillator.

For $k \neq 0$, the operator $T(\mathbf{k})$ only remains invariant against rotations around the vector \mathbf{k} . With this vector taken for the polar axis, factorisation gives the eigenfunctions in the form

$$\phi_{nl\pm m}(\mathbf{k}, \mathbf{v}) = u_{nlm}(k, v, \vartheta) \begin{cases} \cos m\varphi \\ \sin m\varphi \end{cases}, \quad m = 0, 1, 2, \dots, l. \quad (23)$$

Expression (3) yields an eigenvalue equation for u_{nlm} ,

$$(C_m + ikv \cos \vartheta - \lambda_{nlm})u_{nlm} = 0, \quad (24)$$

where the operator C_m differs from (3) by the substitution of $-m^2$ for $\partial^2/\partial\varphi^2$.

Since the choice between $\cos m\varphi$ and $\sin m\varphi$ merely amounts to a rotation around \mathbf{k} , the functions ϕ_{nlm} and ϕ_{nl-m} belong to the same eigenvalue λ_{nlm} . Hence all eigenvalues, except for $m = 0$, are at least double, and this degeneracy is permanent. However, we must expect that in general the m -degeneracy of $\lambda_{nl}(0)$ is lifted by the perturbation $i\mathbf{k} \cdot \mathbf{v}$. The originally $(2l+1)$ -fold eigenvalue $\lambda_{nl}(0)$ may split into $(l+1)$ distinct eigenvalues $\lambda_{nlm}(k)$, one of them simple (λ_{nl0}) and the rest double. The case with constant ζ is again exceptional: the perturbation merely translates the whole spectrum by an amount proportional to k^2 , so that all degeneracies are permanent (McLennan 1966).

When k changes, analytic continuation only affects the factor u_{nlm} in equation (23). If two eigenvalues λ_{nlm} and $\lambda_{n'l'm'}$ with $m \neq m'$ meet, the corresponding eigenfunctions remain distinct. They cannot merge in the way they do at branch points and other singularities of the eigenprojections. Turning the argument around, we see that $m = m'$ is a necessary condition for the eigenprojections associated with two eigenvalues to become singular at the meeting point, e.g. when this is a branch point.

No further rules of this kind have been found, so that only a tentative picture of the general behaviour of the eigenvalues can be drawn (figure 1). It should be noted that beyond the branch point of two eigenvalues λ_{nlm} and $\lambda_{n'l'm}$ the labels n, n', l, l' are no

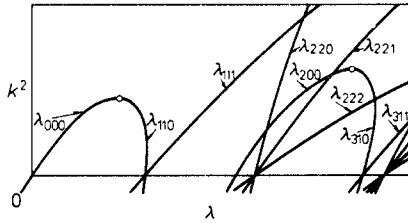


Figure 1. Conceivable behaviour of eigenvalues λ_{nlm} of the Fourier-transformed Fokker-Planck transport operator $T(k)$ in dependence of k^2 . Complex-conjugate pairs of eigenvalues appear beyond the branch points, marked by circles.

longer adequate, and new ones must be invented (n and l cease to be good quantum numbers, one would say, in quantum mechanics).

The same general behaviour of eigenvalues must also be expected with operators $T(k)$ arising from other kinetic equations, in particular from the linearised Boltzmann equation, as long as the eigenvalues remain isolated from essential parts of the spectra. For simple models this can be confirmed by explicit calculation, as this was done for the one-speed model with linearly anisotropic scattering (Davison 1946, Kuščer 1969), where

$$C\phi(\vartheta, \varphi) = \phi(\vartheta, \varphi) - (1/4\pi) \int_{-1}^1 d(\cos \vartheta') \int_0^{2\pi} d\varphi' (1 + b \cos \delta) \phi(\vartheta', \varphi'), \tag{25}$$

$$\cos \delta = \cos \vartheta \cos \vartheta' + \sin \vartheta \sin \vartheta' \cos(\varphi - \varphi').$$

The results for $b = 1$ are shown in figure 2. A branch point appears at $k_0^2 = 0.5738$, where $\lambda(k_0) = 0.5000$.

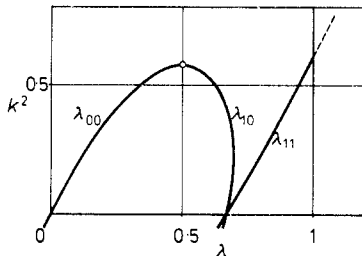


Figure 2. Eigenvalues λ_{lm} of a one-speed model for the Fourier-transformed Boltzmann transport operator. The point $\lambda = 1$ belongs to the essential spectrum, which covers the line $\{\lambda | \lambda = 1 + i k \mu, -1 \leq \mu \leq 1\}$.

7. Comment

The functional-analytic methods used in the present investigation might find useful application also with non-selfadjoint operators occurring in other fields of physics. An example of considerable interest was mentioned by one of the referees: complex Hamiltonians invented for representing resonant states (Moiseyev *et al* 1978). However, the example requires a fresh start, since those operators do not seem to meet all the conditions which facilitated the handling of the present problem.

Acknowledgments

We wish to thank Bernhard Gramsch and Ivan Vidav for encouragement and advice, and the referees for comments which helped to improve the presentation. Part of the investigation was carried out while the first author was at the University of Kaiserslautern, thanks to support by the Deutsche Forschungsgemeinschaft. His contribution to completion of the work (at Ljubljana) was aided by a grant from Raziskovalna skupnost Slovenije.

References

- Corngold N 1977 *Phys. Rev. A* **15** 2454–70
Davison B 1946 *Milne Problem in a Multiplying Medium with Linearly Anisotropic Scattering* (Canadian Atomic Energy Report CRT 358)
Illner R and Kuščer I 1979 *J. Statist. Phys.* **20** 303–16
Kato T 1966 *Perturbation Theory for Linear Operators* (Berlin: Springer)
Kuščer I 1969 *Nucl. Sci. Eng.* **38** 175–6
McLennan J A 1966 *Phys. Fluids* **9** 1581–89
Moiseyev N, Certain P R and Weinhold F 1978 *Molec. Phys.* **36** 1613–30
Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* (New York: McGraw-Hill)